Introduction to topological K-theory

Xuanrui Qi

August 3, 2020

Introduction

This report is an introduction to the K-theory of topological spaces, with a focus on its connection to the K-theory of C*-algebras. We define the K-group of topological spaces and prove the functoriality of the K-theory, and show that the two K-theories are equivalent in some cases.

1 Vector Bundles

Let X be a topological space. To study its K-theory, we need to first consider **complex vector bundles** over it. A vector bundle is a specific type of a *fiber bundle*, which we define hereafter (the definition can be found in almost any text on algebraic topology; here we follow [2]):

Definition 1. Let X be a topological space, and consider the triple $\xi = (E, \pi, X)$, where E is a space and $\pi : E \to X$ a continuous and surjective map.

Let F be another topological space. A *trivialization* of π over an open set $U \subseteq X$ is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

commutes. Here, proj_1 is the canonical projection map. We call the space F the *fiber*, and we say that π is *trivial* over U.

We say that π is *locally trivial* if there exists an open covering $\{U_i\}$ of X such that π is trivial over each $U_i \subseteq X$. We say that the triple $\xi = (E, \pi, X)$

is a ***fiber bundle*** over X if it is locally trivial. By abuse of notation, we may also say that π is a fiber bundle over X.

It is clear that for any $x \in X$, $\pi^{-1}(x)$ is homeomorphic to F, so we also call $E_x = \pi^{-1}(x)$ the *fiber over* x. We call a fiber bundle a *vector bundle* when F is endowed with some vector space structure. There are many different types of vector bundles: in differential geometry, one is interested in C^∞-vector bundles, and in algebraic geometry one is interested in algebraic vector bundles over schemes. In K-theory, however, we are most interested in real or complex vector bundles over plain topological spaces. Here, we will focus on complex vector bundles to illustrate the connection with C*-algebras; most of the theory is completely analogous for real vector bundles. Hereafter, when we say "vector bundle", we will mean a complex vector bundle, which we define below.

Definition 2 ((complex) vector bundle)**.** A complex ***vector bundle*** over a topological space X is a fiber bundle ξ where each fiber $F = \pi^{-1}(X)$ is a finite-dimensional $\mathbb{C}$-vector space, such that the vector space structure on $\pi^{-1}(X)$ is natural, in the sense that the trivialization map φ restricted to x, $\varphi|_{\{x\}}$, induces an isomorphism of vector spaces $\pi^{-1}(x) \to \{x\} \times \mathbb{C}^n \cong \mathbb{C}^n$. If each fiber $\pi^{-1}(x)$ is isomorphic to $\mathbb{C}^n$, then n is called the *dimension* of the vector bundle ξ.[1]

To obtain the definition of a real vector bundle, one may simply replace $\mathbb{C}$ with $\mathbb{R}$.

Remark. It is not necessary that a vector bundle has a fixed dimension, if the base space X is disconnected.

Example 3 (trivial vector bundle)**.** The simplest vector bundle over any topological space X is the triple $\theta_n = (X \times \mathbb{C}^n, \pi, X)$, where π is just the canonical projection map proj_1. Clearly, each fiber has dimension n; we call this vector bundle the *trivial bundle* of dimension n over X.

With an appropriate notion of morphisms, the vector bundles over X form a (small) category.

Lemma 1. *Let X be a topological space. The complex vector bundles over X form a category, where a morphism $(E, \pi, X) \to (F, \rho, X)$ is a continuous map $f : E \to F$ such that the diagram:*

[1] Our definition is a bit different from the definition given in [5], but they are equivalent.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & F \arrow[dl, "\rho"] \\ X & \end{tikzcd}$$

commutes, and such that $f|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \rho^{-1}(x)$ *is a linear map.*

Proof. To show that this is indeed a category, we need to verify that (1) there is an identity morphism, and (2) the composition of morphisms is associative.

We first define the notion of composition of morphisms. Let $\xi = (E, \pi, X)$, $\eta = (F, \rho, X)$ and $\psi = (G, \sigma, X)$ be vector bundles over X, and $f : \xi \to \eta$, $g : \eta \to \psi$ morphisms. Consider the following diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "\pi"'] & F \arrow[r, "g"] \arrow[d, "\rho"] & G \arrow[dl, "\sigma"] \\ & X & \end{tikzcd}$$

and we may define the composition of g with f to be $g \circ f$.

Note that this may cause a slight notional confusion: the composition $g \circ f$ of morphism is defined as the set-theoretic composition of the continuous maps g and f. This is, however, certainly well-defined because the above diagram clearly commutes.

Now, we need to verify that $(g \circ f)|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \sigma^{-1}(x)$ is a linear map: this is, again, clearly true, since the composition of linear maps is again linear. Thus, our notion of composition is well-defined.

The identity morphism is taken as the identity map id, which is certainly continuous; the restriction is again the identity map, which is certainly linear. The associativity of composition is also easy to verify, as it follows from the associativity of function composition. $\square$

Corollary 2. *An isomorphism between two bundles* $\xi = (E, \pi, X)$ *and* $\eta = (F, \rho, X)$ *is a homeomorphism* $h : E \to F$ *such that the diagram*

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "\pi"'] & F \arrow[dl, "\rho"] \\ X & \end{tikzcd}$$

commutes, and such that $h|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \rho^{-1}(x)$ *is an isomorphism of vector spaces. In this case, we call the two vector bundles* ξ *and* η *isomorphic (or* $\xi \cong \eta$*), and denote the isomorphism class of* ξ *as* $\langle \xi \rangle$*. This is indeed an isomorphism in the category of vector bundles.*

We can also define a direct sum operation on vector bundles, analogous to that of C*-algebras.

Definition 4 (direct sum)**.** Let $\xi = (E, \pi, X)$ and $\eta = (F, \rho, X)$ be two vector bundles over X. The ***direct sum*** of ξ and η, or $\xi \oplus \eta$, is defined to be the triple (G, ν, X), where

$$G = \{(v, w) \in E \times F \mid \pi(v) = \rho(w)\},$$

$\nu(v, w) = \pi(v) = \rho(w)$, and where $\nu^{-1}(x) = \pi^{-1}(x) \times \rho^{-1}(x)$ is given the direct sum vector space structure.

Finally, we may also define the *tensor product* of vector bundles (following the definition given in [3]), which does not have a direct analogue in operator K-theory.

Definition 5 (tensor product)**.** Let $\xi = (E, \pi, X)$ and $\eta = (F, \rho, X)$ be two vector bundles over X. We define the ***tensor product*** $\xi \otimes \eta$. as the vector bundle where each fiber over $x \in X$ is the tensor product of the fibers of ξ and η over x. More specifically, we define $\xi \otimes \eta = (G, \sigma, X)$ as the vector bundle with the fiber space defined as the disjoint union

$$G = \sqcup_{x \in X} \pi^{-1}(x) \otimes \rho^{-1}(x),$$

with the topology on G defined below.

For each open set $U \subseteq X$ over which F and G are trivial, we choose two isomorphisms $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$, $i : \rho^{-1}(U) \to U \times \mathbb{C}^m$, where n and m are the dimensions of ξ and η, respectively. The topology $\mathcal{T}_U$ on G is the topology such that each fiberwise tensor product map

$$h \otimes i : \pi^{-1}(U) \otimes \rho^{-1}(U) \to U \times (\mathbb{C}^n \otimes \mathbb{C}^m)$$

is a homeomorphism.

Remark. This construction is well-defined, as we can show that it is independent of the choice of h and i. Consider the continuous maps $f : U \to \mathrm{GL}_n(\mathbb{C})$ and $g : U \to \mathrm{GL}_m(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is the (multiplicative) group of $n \times n$ nonsingular $\mathbb{C}$-matrices (i.e., $\mathbb{C}^n$ automorphisms). By definition any other choice of h and i, say h' and i', can be obtained by composing the second entry with f and g, respectively. That is, if $h(u) = (x, v)$, then we have $h'(u) = (x, f(x)(v))$ for some $f(x) \in \mathrm{GL}_n(\mathbb{C})$.

Now, let us consider $h \otimes i$. We know that $\mathbb{C}^n \otimes \mathbb{C}^m \subseteq \mathbb{C}^{nm}$, so we can apply the aforementioned construction and compose the second entry with $f \otimes g$, which are a family of continuous maps $U \to \mathrm{GL}_{mn}(\mathbb{C})$.

Then, if we restrict our construction to an open subset $V \subseteq U$, we can see that the subset topology of V in U is the same as the topology $\mathcal{T}_V$, as local trivializations restrict to local trivializations on open subsets. Therefore our construction is well-defined, independent of the choice of h and i.

2 The K-group of vector bundles

To obtain the K-theory of topological spaces (via vector bundles), we need to form Grothendieck groups out of vector bundles. Just like in operator K-theory, we apply the Grothendieck construction to an abelian semigroup, which we define below.

Proposition 3. *The set of all isomorphism classes of vector bundles ξ over a topological space X forms a abelian semigroup, where addition is defined as $\langle \xi \rangle + \langle \eta \rangle = \langle \xi \oplus \eta \rangle$. We denote this abelian semigroup by* $\mathbf{Vect}(X)$.

Proof. We need to show that addition is associative and commutative.

To prove that addition is associative, we need to show that $(\langle \xi \rangle + \langle \eta \rangle) + \langle \psi \rangle = \langle \xi \rangle + (\langle \eta \rangle + \langle \psi \rangle)$, i.e., $\langle (\xi \oplus \eta) \oplus \psi \rangle = \langle \xi \oplus (\eta \oplus \psi) \rangle$. This boils down to the fact that the direct sum of vector spaces is associative up to isomorphism.

The commutativity of addition follows from the fact that the direct sum of vector spaces is commutative. □

Taking the Grothendieck construction on $\mathbf{Vect}(X)$, we obtain the Grothendieck group $K^0(X) = \mathcal{G}(\mathbf{Vect}(X))$. Sometimes, we also write $K(X)$ instead of $K^0(X)$. We often write the image of $\langle \xi \rangle$ under the Grothendieck map, $\gamma : \mathbf{Vect}(X) \to K^0(X)$, $\gamma(\langle \xi \rangle)$, as $[\xi]$.

However, unlike the Grothendieck group of a C*-algebra, the Grothendieck group of a topological space is not only an abelian group but also a commutative ring, inheriting its multiplication operation from $\mathbf{Vect}(X)$, which we proceed to define below.

Definition 6. We define multiplication on $\mathbf{Vect}(X)$ as $\langle \xi \rangle \cdot \langle \eta \rangle = \langle \xi \otimes \eta \rangle$.

Corollary 4. $\mathbf{Vect}(X)$ *forms a commutative semi-ring under the addition and multiplication operations defined above.*

Proof. To show that multiplication is commutative, it suffices to show that the tensor product of vector bundles is symmetric up to isomorphism. Let $\xi = (E, \pi, X)$ and $\eta = (F, \rho, X)$ be vector bundles over X. Given any $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$ and $i : \rho^{-1}(U) \to U \times \mathbb{C}^m$, we have:

$$h \otimes i : \pi^{-1}(U) \otimes \rho^{-1}(U) \to U \times (\mathbb{C}^n \otimes \mathbb{C}^m)$$

and

$$i \otimes h : \rho^{-1}(U) \otimes \pi^{-1}(U) \to U \times (\mathbb{C}^m \otimes \mathbb{C}^n).$$

The domain and codomain are isomorphic (this follows from the symmetry of the tensor product of vector spaces), and the two maps are equivalent, since the entries of $h \otimes i$ (and $i \otimes h$), represented as a matrix, are simply the product of the entries of h and i. Thus we have $h \otimes i \cong i \otimes h$. This follows that $\xi \otimes \eta \cong \eta \otimes \xi$, and thus $\langle \xi \rangle \cdot \langle \eta \rangle = \langle \eta \rangle \cdot \langle \xi \rangle$.

Next, we need to show that multiplication is associative, and this is an easy consequence of the fact that the tensor product of vector spaces is associative.

Finally, we need to show that multiplication distributes over addition. Again, this is a direct consequence of the fact that the tensor product of vector spaces distributes over the direct sum. □

Definition 7. Let $(\langle \xi_1 \rangle, \langle \eta_1 \rangle)$ and $(\langle \xi_2 \rangle, \langle \eta_2 \rangle)$ be isomorphism classes in $K(X) = \mathcal{G}(\mathbf{Vect}(X))$. We define the multiplicative operation on $K(X)$ as

$$(\langle \xi_1 \rangle, \langle \eta_1 \rangle) \cdot (\langle \xi_2 \rangle, \langle \eta_2 \rangle) = (\langle (\xi_1 \otimes \xi_2) \oplus (\eta_1 \otimes \eta_2) \rangle, \langle (\xi_1 \otimes \eta_2) \oplus (\xi_2 \otimes \eta_1) \rangle).$$

It is routine to verify that this operation is associative, commutative and distributive over addition [4].

Under this multiplicative operation, $K(X)$ forms a commutative ring, which is also called the *K-theory ring* of X, where the multiplicative identity element is θ_1, the trivial vector bundle over X of dimension 1.

3 The K-functor

Just like in operator K-theory, $K(-) = K^0(-)$ is a functor to the category **Ab** of abelian groups. In fact, $K(-)$ is a functor from the category **Top** of topological spaces to the category **CRing** of commutative rings. However, unlike the K_0 functor in operator K-theory, which is covariant, our functor K^0 is contravariant.

In order to prove the functoriality of K^0, we need to first define the notion of the *pullback* of vector bundles.

Definition 8. Let X, Y be topological spaces and $f : X \to Y$ a continuous map. Let $\eta = (F, \rho, Y)$ be a vector bundle over Y. Then we define the ***pullback*** of η along f, or $f_*\eta$, to be the triple (E, π, X), where

$$E = \{(x, v) \in X \times F \mid f(x) = \rho(v)\},$$

such that $\pi(x, v) = x$, and where the fiber $\pi^{-1}(x) = \rho^{-1}(f(x))$ inherits its vector space structure from η.

Remark. For the categorically minded, the pullback of $f_*\eta$ is nothing other than the category-theoretic pullback of ρ along f:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & F \arrow[d, "\rho"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If the diagram above is a pullback square, then (E, π, X) is the pullback of ρ along f.

Naturally, this pullback induces a pullback map $f_* : \mathbf{Vect}(Y) \to \mathbf{Vect}(X)$, mapping $\langle\eta\rangle$ to $\langle f_*\eta\rangle$. This pullback map is compatible with both the direct sum and the tensor product of vector bundles.

Lemma 5. *The pullback map is compatible with addition and multiplication on* $\mathbf{Vect}(Y)$*, that is:*

(1) $f_*(\langle\eta\rangle + \langle\psi\rangle) = f_*(\langle\eta\rangle) + f_*(\langle\psi\rangle)$*;*

(2) $f_*(\langle\eta\rangle \cdot \langle\psi\rangle) = f_*(\langle\eta\rangle) \cdot f_*(\langle\psi\rangle)$*.*

Proof. Showing (1) is equivalent to showing that $f_*(\eta \oplus \psi) \cong f_*\eta \oplus f_*\psi$. Showing (2) is equivalent to showing that $f_*(\eta \otimes \psi) \cong f_*\eta \otimes f_*\psi$. Both can be shown by a diagram chase. $\square$

Next, we may extend this pullback construction to $K(X)$. Define the map $f_* : K(Y) \to K(X)$ (note that we are abusing notation here) by $f_*([\eta]) = [f_*\eta]$. Again, one can verify that f_* is compatible with addition and multiplication in $K(Y)$, and thus f_* is a ring homomorphism. This brings us to our first main theorem:

Theorem 6 (functoriality of K^0). *Any continuous map of topological spaces $f : X \to Y$ induces a ring homomorphism $K^0(f) = f_* : K^0(Y) \to K^0(X)$, such that the diagram*

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \\ K^0(Y) \arrow[u, leftarrow] \arrow[r, "K^0(f)"'] & K^0(X) \arrow[u, leftarrow] \end{tikzcd}$$

commutes.

In other words, $K^0 : \textsf{Top}^{\mathrm{op}} \to \textsf{CRing}$ is a contravariant functor from the category of topological spaces to the category of commutative rings, sending each object (i.e., topological space) X in $\textsf{Top}$ to its K-theory ring $K^0(X)$, and each morphism (i.e. continuous map) $f : X \to Y$ to the pullback map $f_* : K^0(Y) \to K^0(X)$.

4 Connections to the K-theory of C*-algebras

Hereafter, we will assume that the base space X is compact and Hausdorff. For the sake of simplicity, we will also assume that X is connected (so that bundles over X have fixed dimension). Then, we know that $C(X)$, the vector space of $\mathbb{C}$-valued functions on X[2], is a unital C*-algebra. We wish to study the relationship between X and $C(X)$ from a K-theory point of view. Naturally, the big question we are interested is the relationship between their respective K-groups, $K^0(X)$ and $K_0(C(X))$. It turns out that they are in fact isomorphic as abelian groups.

Theorem 7. *Let X be a compact, connected and Hausdorff topological space. Then $\mathcal{D}(C(X)) \cong \mathbf{Vect}(X)$ as abelian semigroups. Furthermore, we have $K^0(X) \cong K_0(C(X))$ (as abelian groups).*

Proof. For each projection $p \in \mathcal{P}_\infty(C(X))$, we can associate a vector bundle $\xi_p = (E_p, \pi, X)$ over X in the following way. Let

$$E_p = \{(x, v) \in X \times \mathbb{C}^n \mid v \in p(x)(\mathbb{C}^n)\},$$

noting that $M_n(C(X)) = C(X, M_n(\mathbb{C}))$. Set $\pi(x, v) = x$, and give the fiber $\pi^{-1}(x) = p(x)(\mathbb{C}^n)$ the vector space inherited from $\mathbb{C}^n$, i.e., such that the fiber is a subspace of $\mathbb{C}^n$.

[2]Since X is compact, we have $C(X) = C_0(X) = C_b(X)$.

One can verify that $\xi_p \cong \xi_q$ if and only if $p \sim_0 q$ for any $p, q \in \mathcal{P}_\infty(C(X))$, and that $\xi_{p \oplus q} \cong \xi_p \oplus \xi_q$.

Now, consider the correspondence map $V : \mathcal{D}(C(X)) \to \mathbf{Vect}(X)$, defined as $[p]_\mathcal{D} \mapsto \langle \xi_p \rangle$. By the correspondence of $\cong$ with $\sim_0$, we know that this map is well-defined and injective. Moreover, we know that this map is additive (i.e., $V([p]_\mathcal{D} + [q]_\mathcal{D}) = \langle \xi_p \rangle + \langle \xi_q \rangle$).

To show that V is an isomorphism, however, we will also need to show that it is surjective. Here, we will need a theorem [6] originally proposed by J.-P. Serre and stated in present form by R. Swan. $\square$

Theorem 8 (Serre-Swan)**.** *For each vector bundle ξ over a compact and Hausdorff space X, there is another bundle η over X, such that $\xi \oplus \eta$ is the trivial bundle, i.e., $\xi \oplus \eta \cong \theta_n$ for some n.*

Proof. The proof can be found in either [6], or in a more modern form in [3]. $\square$

Now we can return to the proof of Theorem 7.

Proof of Theorem 7 (continued). Let $\xi = (E, \pi, X)$ be any bundle over X. We can show that there is a $p \in \mathcal{P}_\infty(C(X))$ such that $V([p]_\mathcal{D}) = \xi$. By Serre-Swan, we know that there is a bundle $\eta = (F, \rho, X)$ such that $\xi \oplus \eta = \theta_n$ for some n. In other words, we know that for each respective fiber over x, we have $\pi^{-1}(x) \oplus \rho^{-1}(x) \cong \mathbb{C}^n$. Then we can take $p(x)$ as the canonical projection map $\mathbb{C}^n \to \pi^{-1}(x)$, and as such $p : X \to M_n(\mathbb{C})$ is a continuous map, i.e. $p \in \mathcal{P}_n(C(X))$. Moreover, we have $\xi_p = \xi$. Therefore, we may conclude that V is surjective. Since V is bijective and additive, it is an isomorphism of semigroups. It follows immediately that $K_0(C(X))$ and $K^0(X)$ are isomorphic as abelian semigroups. $\square$

For the categorically minded reader, we may intepret Theorem 7 as a result about functors. Consider the (contravariant) functor $C(-) : \mathsf{CH}^{\mathrm{op}} \to$ **C*Alg** from the category of compact Hausdorff spaces to the category **C*Alg** of C*-algebras. This functor takes each space X to the C*-algebra of functions $C \to \mathbb{C}$, and each continuous map $f : X \to Y$ to a $*$-homomorphism $f_* : C(Y) \to C(X)$ defined by $f_* h = h \circ f$. It is not hard to verify that this is indeed a functor.

Since **CH** is a full subcategory of **Top**, we can restrict K^0 (and ignore the multiplicative structure on the K-theory ring) to a functor $\mathsf{CH} \to \mathsf{Ab}$. Theorem 7 means that the functors K^0 and $K_0 \circ C(-)$ are naturally isomorphic as functors $\mathsf{CH}^{\mathrm{op}} \to \mathsf{Ab}$.

Thus, we see that the K-theory of topological spaces and of (complex) C*-algebras are closely connected. Nevertheless, operator K-theory is in a sense more "general" than topological K-theory, in the sense that commutativity is not required (hence the name "non-commutative geometry"), and as such there is no natural ring structure on $K_0(\mathcal{C})$, unless $\mathcal{C} = C(X)$ for some compact Hausdorff space X.

5 Further Topics

The K-theory of topological spaces yield many important topological invariants that could be used to study the property of those spaces. The Chern character is one of them: it is a ring homomorphism $\mathrm{ch} : K^0(X) \to H^{\mathrm{ev}}(X; \mathbb{Q})$ from the K-theory ring of X into the even cohomology ring of X. The Chern character is exactly the equivalent of the cyclic cohomology of C*-algebras in topology, but the theory takes a lot to develop, so we refer readers to either [3], or [1] for a perspective leaning towards differential geometry.

6 Conclusion

We have introduced the basics of topological K-theory, defined the K-theory ring $K(X)$ of a topological space X, and proved the functoriality of the K-functor K^0, in a development analogous to the K-theory of C*-algebras. Furthermore, we have also outlined the close connection between topological K-theory and operator K-theory, and proved the equivalence of the two K-theories when X is compact Hausdorff.

References

[1] Raoul Bott and Loring Tu. *Differential Forms in Algebraic Topology.* Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, 1982.

[2] Tammo tom Dieck. *Algberaic Topology.* Zürich: European Mathematical Society, 2008.

[3] Allen Hatcher. *Vector Bundles and K-Theory.* 2017. URL: `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`.

[4] nLab authors. *Topological K-theory.* Revision 104. Aug. 2020. URL: `http://ncatlab.org/nlab/show/topological%5C%20K-theory`.

[5] M. Rørdam, F. Larsen, and N. Laustsen. *An Introduction to K-Theory for C*-Algebras.* Vol. 49. London Mathematical Society Student Texts. Cambridge University Press, 2000.

[6] Richard G. Swan. "Vector bundles and projective modules". In: *Transactions of the American Mathematical Society* 105 (1962). URL: https://www.ams.org/journals/tran/1962-105-02/S0002-9947-1962-0143225-6/S0002-9947-1962-0143225-6.pdf.