**２．【研究計画】**※適宜概念図を用いるなどして、わかりやすく記入してください。なお、本項目は１頁に収めてください。様式の変更・追加は不可。

**(1) 研究の位置づけ**

特別研究員として取り組む研究の位置づけについて、当該分野の状況や課題等の背景、並びに本研究計画の着想に至った経緯も含めて記入してください。

### (1) Positioning of my research

The big picture behind my proposal is the so-called "homotopy-coherent" mathematics that has its origins in Grothendieck-style algebraic geometry, but has flourished in the $21^{st}$ century: the most prominent examples would be homotopy theory and Lurie's derived algebraic geometry [Lur04]. My proposed research is also based on an existing line of research on **synthetic** approaches to mathematics. Synthetic mathematics, or "axiomatic" mathematics done by imposing axioms and conditions on mathematical objects, is best exemplified by Euclidean geometry, but has since been extended much further to e.g. differential geometry.

### (2) Background

For this new quickly growing line of research, many people believed that traditional foundational theories were not adequate: as such, Voevodsky et al. proposed **homotopy type theory (HoTT)**, a new foundation based on Martin-Löf's type theory. HoTT allows convenient treatment of "higher" structures such as $\infty$-groupoids and categories, which make it highly attractive for homotopy-coherent mathematics. Meanwhile, **synthetic mathematics** has a close connection to category theory and type theory, as type theory is a natural framework for such mathematics.

### (3) Towards this research proposal

The aforementioned approach has already been proven to be rather fruitful in algebraic topology, but it has not yet been widely used in other areas of mathematics. One reason is that HoTT models algebraic topology very well, but for other branches of mathematics, pure HoTT does not make the cut because one generally wants to work in special categories — so **modalities** are required to capture the "nice" properties of those categories. For example, in synthetic differential geometry, one might want to work in a smooth or cohesive topos; in synthetic algebraic geometry, one wants to work in Zariski toposes, étale toposes, or even algebraic stacks. Modal type theories need to be developed for these applications, and more work is required to further develop actual mathematics in these type theories. There is some prior work towards this approach [Wel17, SS14], but there is still much unexplored potential. My proposed research will extend these prior work to other areas of mathematics, particularly (algebraic and differential) geometry.

参考文献

[Lur04] Jacob Lurie, *Derived Algebraic Geometry.*

[Wel17] Felix Wellen, *Formalizing Cartan Geometry in Modal Homotopy Type Theory.*

[SS14] Urs Schreiber and Michael Shulman, "Quantum gauge field theory in cohesive homotopy type theory," arXiv:1408.0054.

**【研究計画】（続き）** ※適宜概念図を用いるなどして、わかりやすく記入してください。なお、各事項の字数制限はありませんが、全体で２頁に収めてください。様式の変更・追加は不可。

**(2) 研究目的・内容等**

① 特別研究員として取り組む研究計画における研究目的、研究方法、研究内容について記入してください。
② どのような計画で、何を、どこまで明らかにしようとするのか、具体的に記入してください。
③ 研究の特色・独創的な点（先行研究等との比較、本研究の完成時に予想されるインパクト、将来の見通し等）にも触れて記入してください。
④ 研究計画が所属研究室としての研究活動の一部と位置づけられる場合は申請者が担当する部分を明らかにしてください。
⑤ 研究計画の期間中に受入研究機関と異なる研究機関（外国の研究機関等を含む。）において研究に従事することも計画している場合は、具体的に記入してください。

## 2.1 Goals

The major goal of my proposed research project is to make progress towards a new foundation for mathematics based on **modal type theory**, and investigate how to do mathematics **synthetically** in this foundational theory. There are many benefits to using type theory as opposed to conventional foundations like ZF(C), but the most important benefit is perhaps that type theory makes it easy to write **machine-checkable proofs** which can help eliminate errors and gaps in proofs. Towards this goal, three smaller goals can be identified: (1) the development of modal type theories appropriate for certain mathematical applications, (2) the development of synthetic mathematics, such as differential and algebraic geometry, in these type theories, (3) the development of a proof assistant, or a program to check the correctness of proofs, for the said type theory.

A secondary goal of my research is to use the obtained modal type theory not only as a foundation of mathematics, but also a basis for computation, as type theories are inherently programming languages. If time permits, I would like to (4) explore the potential of developing a programming language for synthetic differential geometry. This language could be used for differential computation and has potential applications in machine learning.

## 2.2 Outline of research

The goals (1) and (2) are to be carried out in conjunction, as it would be impossible to evaluate the aptness of a modal type theory as a foundation of an area of mathematics without actually carrying out the mathematics in it. Once there is some progress on (1) and (2), I will start working towards on (3), but (1) and (2) are the main priorities. (4) would only be considered if goals (1)–(3) progressed significantly faster than expected.

## 2.3 Content of research

(1) Developing modal type theories

(2) Synthetic mathematics in modal type theories

The first part of my research project will be to convert existing work in synthetic mathematics to a modal type theoretic setting. A particular goal is synthetic algebraic geometry [Ble17], as there are still many unresolved problems. Moreover, algebraic geometry can be extended to the higher situation (via algebraic stacks and derived schemes), which can better highlight the usefulness of type-theoretic foundations. Accordingly, the type theory used would be a variety of HoTT extended with modalities, following [RSS17].

- Type theories for algebraic geometry. [Ble17] develops algebraic geometry synthetically in the small and big Zariski toposes, and their internal logics are modal type theories. However, these are not the only toposes we want to work in; what would a type theory for e.g. the small/big étale topos look like? Furthermore, since it is not effective to have different type theories for each topos we want to work in, I will study potential connections to cohesive type theory [SS14], which has been proposed as a basis synthetic geometry in general.

- Further development of synthetic algebraic geometry. Once we have a setting to do algebraic geometry, we may want to check if it is indeed useful by actually doing some algebraic geometry in it. The primary goal at the moment is to develop a synthetic treatment of sheaf cohomology; this will be carried out in collaboration with algebraic geometers. The process of attempting to formulate such a theory will shed light on what is desirable and what is not in a modal type theory serving as a foundation for algebraic geometry.

- Connections to synthetic differential geometry [Koc99]. Synthetic differential geometry (SDG) is a subject that is much more well developed than synthetic algebraic geometry. However, the connections between the two subjects need to be elucidated. It is well known mathematical folklore that modern algebraic geometry is often inspired by differential geometric concepts, and working in a modal type theory may help us formalize this insight and better understand how the two subjects are connected and where they differ.

(3) Development of proof assistant (4) From modal TT to programming language

Once progress has been made towards (1) and (2), it would be possible to start working on a proof assistant for the modal type theory we have developed.

With the experience gained in (1)–(3), I will attempt to develop a programming language based on SDG which can describe differentiable computation, a model of computing where every function can be differentiated. The semantics will be described by certain higher models of SDG to determined. Practical questions that may arise—particularly those related to effectful computation—will be studied in collaboration with computer scientists.

### 2.4 Impact and innovation

This project will bring about two important innovations: the treatment of algebraic geometry in a fully constructive, type-theoretic framework, and the development of a proof assistant based on modal dependent type theory. To the best of the author's knowledge, both are completely novel goals. The former would deeply renovate our current understanding of algebraic geometry especially in the higher case, and has some direct implications towards computational algebraic geometry. The later would bring us new insight on the development of proof assistants, and the development of a tactic language for modal TT would be especially valuable.

参考文献

[Koc99] Anders Kock, *Synthetic Differential Geometry.*

[Ble17] Ingo Blechschmidt, *Using the Internal Language of Toposes in Algebraic Geometry.*

[RSS17] Egbert Rijke, Michael Shulman, and Bas Spitters, "Modalities in homotopy type theory," arXiv:1706.07526.

## ３．人権の保護及び法令等の遵守への対応 ※本項目は１頁に収めてください。様式の変更・追加は不可。

本欄には、「２．研究計画」を遂行するにあたって、相手方の同意・協力を必要とする研究、個人情報の取り扱いの配慮を必要とする研究、生命倫理・安全対策に対する取組を必要とする研究など法令等に基づく手続が必要な研究が含まれている場合に、どのような対策と措置を講じるのか記入してください。例えば、個人情報を伴うアンケート調査・インタビュー調査、国内外の文化遺産の調査等、提供を受けた試料の使用、侵襲性を伴う研究、ヒト遺伝子解析研究、遺伝子組換え実験、動物実験など、研究機関内外の情報委員会や倫理委員会等における承認手続が必要となる調査・研究・実験などが対象となりますので手続の状況も具体的に記入してください。

なお、該当しない場合には、その旨記入してください。

Not applicable.

**４．【研究遂行力の自己分析】**※各事項の字数制限はありませんが、全体で２頁に収めてください。様式の変更・追加は不可。
本申請書記載の研究計画を含め、当該分野における(1)「研究に関する自身の強み」及び(2)「今後研究者として更なる発展のため必要と考えている要素」のそれぞれについて、これまで携わった研究活動における経験などを踏まえ、具体的に記入してください。

**(1) Strengths as a researcher**

I believe that my main strength is that I come from an interdisciplinary background. I come from a background in theoretical computer science, with bachelor's and master's degrees in computer science. However, I also received formal mathematical training during my study at the Graduate School of Mathematics, Nagoya University, with a special focus on category theory and related topics.

Prior research experience

In the particular area of types and theorem provers, I already have some research experience. My past research is mostly related to design issues around type theory and/or applications of theorem provers, which are both topics closely relevant to my proposed research. I have one published, peer-reviewed paper published in a selective computer science conference (in computer science, most work is published in conferences rather than journals):

1. "Proving Tree Algorithms for Succinct Data Structures", R. Affeldt, J. Garrigue, X. Qi, K. Tanaka, ITP 2019, published as LIPIcs **141**, 5:1-5:19 (2019).

I have also given a few presentations at important international and domestic workshops. Except when noted, all the following talks are peer-reviewed:

1. "Experience Report: Type-Driven Development of Certified Tree Algorithms in Coq", R. Affeldt, J. Garrigue, X. Qi, K. Tanaka, presented at the Coq Workshop 2019.
2. "Rings, Categories and Schemes in Coq", X. Qi, presented at the Theorem Proving and Provers Meeting 2020 (domestic workshop, non-peer reviewed).
3. "Towards a Coq Specification for Generalized Algebraic Datatypes in OCaml", X. Qi, presented at the CoqPL Workshop 2021.

My research project is rooted in type theory, but it also involves some serious algebraic geometry. Therefore, with training in and experience with both areas, I believe that I am in a position to complete this research. In particular, I have already done some work combining the two areas, on the formalization of "textbook-style" algebraic geometry (see presentation 2) in a type theory-based theorem prover, which in part led me to the realization that a new foundation and synthetic methods are necessary.

Practical considerations

Type theory and foundations of mathematics is not a purely theoretical subject; many non-theoretical considerations exist. For example, in developing foundations of mathematics, the *usability* of such a foundation is often neglected. For my project, in particular, the ease of writing abstract proofs in the type theory, and of implementing the type theory as a proof assistant, is extremely important. This involves many user experience issues that could not be answered purely by theory. Having completed several projects which involve heavily formalization work in proof assistants (all the results listed above involved formalization in Coq), I believe that I am in a position to make well-informed decisions on the design aspects of the proof assistant to be developed. Furthermore, I have experience with language design (see presentation 3), as well as significant software engineering experience both in industry and in research; both are essential for carrying out the implementation work outlined in my proposal.

**(2) Elements required for further development as a researcher**

## Development as pure mathematics researcher

Although I already have significant research experience, on both theoretical and more practical aspects in computer science, I do not yet have much experience with the more purely mathematical aspects of my proposed research project. Compared to research in computer science which is often quite goal-driven, mathematical research can be much more open-ended. Although I have proposed some specific goals, such as the synthetic development of sheaf cohomology, which would guide my research, the potential paths to attain those goals remain unknown; therefore, much exploration is required. To further develop my abilities to conduct such exploratory research, I plan to present my work frequently towards algebraic geometry audiences, by e.g. giving talks at colloquiums and/or informal algebraic geometry workshops. I believe that the feedback gained from experts of algebraic geometry will help guide the exploration process, and also enhance my development as a mathematics researcher.

## A Broader Perspective

Despite that I already come from an interdisciplinary background, it is always important to further broaden my perspective given my proposed project is highly interdisciplinary in nature. Specifically, I am interested in potential applications to physics, specifically the areas of gauge theory and quantum field theory (e.g. [SS14]); there may also be other applications towards other parts of mathematics, such as differential geometry and analysis. To gain a perspective towards these subjects and the potential connections between my work and these areas, I plan to organize seminars and workshops on topos theory, homotopy type theory, and related topics, open to participants working in multiple areas. This will not only broaden perspectives, but also foster interdisciplinary collaborations, and help me gain personal and leadership skills essential for success in academia.

**５．【目指す研究者像等】**※各事項の字数制限はありませんが、全体で１頁に収めてください。様式の変更・追加は不可

日本学術振興会特別研究員制度は、我が国の学術研究の将来を担う創造性に富んだ研究者の養成・確保に資することを目的としています。この目的に鑑み、(1)「目指す研究者像」、(2)「目指す研究者像に向けて特別研究員の採用期間中に行う研究活動の位置づけ」を記入してください。

### (1) Image of researcher I aim to be

I aim to be a research mathematician but with an interdisciplinary outlook. I came from a multidisciplinary background myself, and I believe that even for pure mathematicians, an interdisciplinary perspective is extremely useful, as it could widen one's eyesight and provide important insights. Therefore, I aim to become a "bridge" between researchers in different areas. I aim to make my research relatable and understandable to researchers in a range of areas in order to foster new connections between different subjects. I also plan to collaborate extensively with researchers in areas other than mathematics and logic, such as those in computer science, physics, and even philosophy of science.

Moreover, I aim to be a theoretical researcher with an eye on the real world. Despite that my research interests are decidedly theoretical, I believe that an eye on applied subjects can provide an important perspective towards my research. Moreover, since my research area has close connections to practical aspects of computer science, I aim to collaborate with computer scientists to put some of my ideas into fruition.

### (2) Research activities towards the aforementioned goal

My research project, as described before, naturally requires many interdisciplinary collaborations. For example, for developing usable type theories, collaboration with computer scientists will be required; for the technical content of algebraic geometry, I expect much collaboration with algebraic geometers. Naturally, this will help me develop my interdisciplinary outlook and train my ability to work with people of different academic backgrounds. Moreover, since my project also has a small applied portion in the development of a proof assistant (and potentially also of a programming language), this will offer me opportunities to connect with applied researchers and, and also provide me a unique chance for me to consider how to connect a seemingly very theoretical project with practical goals.

Especially, since my project will potentially involve deep algebraic geometry content, long-term working relationships with algebraic geometers will be especially useful, as the technical details of algebraic geometry can be daunting and could be only completely understood through continuous collaborations and discussions. I consider my project belonging mainly to type theory and foundations of mathematics, and I plan to target mainly journals and/or conferences in logic and/or theoretical computer science. However, I also plan to present my work at algebraic geometry workshops and conferences in order to have in-depth academic exchanges.

Moreover, to foster connections between researchers in different areas, I plan to host a series of seminars and informal workshops on topics related to my work, particularly topos theory and homotopy type theory. I hope to target not only researchers who are working on "inter-area" topics like me, but also researchers who are working in their own area who are curious. Through talks and discussions, I will try to elucidate the value of my work to researchers working in areas that are different from mine. I also hope to, if time permits, write some tutorials and expositions for topos theory and homotopy type theory, targeting mainly computer scientists and mathematicians working in areas more distant from mine.