# Curves in synthetic algebraic geometry: approaches, open problems and challenges

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# Disclaimer

- I'm a PhD student at the Graduate School of Mathematics, Nagoya University in Japan. My interests are in categorical logic and topos/sheaf theory.
- I am no algebraic geometer, and my familiarity with AG is really just mediocre at best.
- If you spot a mathematical error, do correct me immediately!
- This project can go towards many, many different directions, and I'm not sure what this leads to...

# Why synthetic curves?

- Most basic examples in algebraic geometry, long known to algebraic geometers
- Simplest nontrivial case of algebraic varieties
- Easy to classify: classification depends only on genus
- Nevertheless requires a considerable amount of machinery
- A great testbed for ideas

# What is a curve?

Usually: an algebraic variety (*an irreducible, reduced scheme, separated and of finite type over a field*) of dimension 1.
**What do all of those mean?**

# Unfolding the definition

irreducible, reduced, of finite type: Similar to their usual meanings. Easy!

. . . separated. . .: Not totally sure a definition similar to the one used externally will work, but it possibly does!

. . . of dimension 1: Every scheme is a formal manifold in the sense of SDG, so just use the releavant notion from SDG! Question: is that right?

. . . over a field: This is the hard part! Any “conventional” approach is known to not work.

# Problem 1: working over fields

- Problem: algebraic varieties are always defined over fields. In SAG, the base ring $R$ is the internal view of $\mathbb{A}^1_S$ ($\operatorname{Spec} k[t]$ in the case of scheme over a field).
- $R$ is *always* a constructive field, but it is impossible to use axioms to model "$R$ is externally $\operatorname{Spec} k[t]$".
- Solution 1: use a modality ("specific algebraic geometry").
- Solution 2: work in the fppf topos?
- Solution 3: ignore it, until we get stuck.

# Riemann-Roch

Classically stated:

## Riemann-Roch theorem

For any (Cartier) divisor on a curve $C$, there exists a $g : \mathbb{N}$ (called the *genus* of $C$) such that:

$$\ell(D) - \ell(K - D) = \deg D + 1 - g$$

where:

- $\ell(D)$ is the dimension of $\mathcal{L}(D)$, the space of global sections of the line bundle associated to the divisor $D$;
- $\deg D$ is the degree of $D$;
- $K$ is $C$'s *canonical divisor*.

**A lot to work out!**

# First challenge: divisors

**Weil or Cartier?** For curves, usually Weil divisors are much more straightforward to work with. For general varities, Cartier divisors can actually be easier.

**A caveat about Weil divisors** The conventional definition as a linear combination of codimension 1 subvarities is perhaps not very useful. Instead, it's better to understand it as a function that maps subvarities to an integer (degree of vanishing).

**Cartier divisors** Cartier divisors, on the other hands, could be actually simpler—just internalize the relevant sheaf theoretic statements!

# De-sheafifying definitions

- From the internal point of view, a sheaf is just a "type equipped with some open covering of a space"!
- A Cartier divisor is a global section of $\mathcal{M}^\times/\mathcal{O}^\times$...
- or synthetically, just a type of (local) rational functions on $X$ quotiented by multiplication of functions from a open neighborhood of $X$ to $R^\times$!
- if the local functions are not only rational but regular (i.e., identically defined on all of $U$) then the divisor is called *effective*...
- Would "holomorphic" and "meromorphic" be better terminology in this case?

# Divisors, continued

- Effective Cartier divisors are often defined in terms of ideal sheaves of subschemes too. Can we reconstruct the correspondence synthetically?
- Weil divisors, on the other hand, can be tricky synthetically. One can define them as functions from the set of codimension-1 subvarieties to $\mathbb{Z}$. . .
- in the case of a curve, this is just a function from the curve $C$ itself to $\mathbb{Z}$. The case of Riemann surfaces give a good intuition.
- OTOH, we probably can't go from linear combinations to this function. . . But surely we can go the other way?

# Divisors, finally

- Cartier divisors have a correspondence to line bundles, and in the case of curves we can use that to define the degree of divisors...
- or use one of many other approaches. (Don't ask me, though, because I don't quite understand the details either.)
- Now we have $\ell(-)$ and $\deg D$ defined, and can proceed to state Riemann-Roch!

# Proving Riemann-Roch

- There are many existing proofs of Riemann-Roch:
- the most well-known one uses Serre duality, which relies on homological techniques and is unlikely to work well for us;
- there is an simple algebraic proof which may be helpful;
- there is also the complex analytic proof: uses analytic tools that we have no access to, but might give us some interesting ideas nontheless.
- This is the part we know least about! What is the most promising path to Riemann-Roch?

# From curves to surfaces

- Once Riemann-Roch is proven, we can basically use it to classify all projective curves.
- Naturally, we move towards surfaces. . . which are way more complicated.
- Interesting ideas: blowing up, basic birational geometry, K3 surfaces, 27 lines, etc.
- Disclaimer: I know close to nothing about surfaces. . .

# Connecting with complex geometry?

- Of course, there is no synthetic complex geometry as of yet.
- But I see how a synthetic approach would be very useful for complex geometry/multiple complex variables!
- Can we prove a synthetic correspondence between algebraic and complex geometry?